THE STABILITY OF A PERIODIC SOLUTION OF THE NAVIER-STOKES EQUATION

(OB USTOICHIVOSTI ODNOGO PERIODICHESKOGO RESHENIIA URAVNENIIA NAV'E-STOKSA)

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A two-dimensional problem for the Navier-Stokes equation is considered and the stream function is assumed to be periodic in x, y and t, the periods being $2\pi/\alpha_0$, 2π and T respectively.

A steady state solution $\psi(y) = -(\gamma/\nu)\cos\gamma$ was investigated in [1] where it was shown that the solution is stable when $\alpha_0 > 1$ and unstable when $\alpha_0 < 1$ and ν is sufficiently small. Below we investigate the stability of a periodic solution

$$\psi_0(y, t) = -\frac{\gamma}{\nu}\cos y\,(1 + \varepsilon \sin \omega t) \qquad \left(T = \frac{2\pi}{\omega}\right)$$

which differs little from the steady state solution (ϵ is a sufficiently small positive number). Also, $\sin \omega t$ can be replaced by any function $g(t)$ periodic and of period T and such, that

$$\int_0^T g(t)\,dt = 0$$

At the beginning of [1] it was remarked that the periodic solution $\psi_0(y, t)$ is unconditionally stable when $\alpha_0 \geq 1$. Proof of the above statement is analogous to that found in [2] for the steady solution.

The main result of this work consists of the proof that when $\alpha_0 < 1$, $\lambda = \gamma/\nu^2$ is sufficiently large and ϵ is sufficiently small, then the solution $\psi_0(y, t)$ is unstable with respect to infinitesimal perturbations.

When investigating the spectrum of stability, we have found that simple eigenvalues were obtained in the range $\frac{1}{2} \leq \alpha_0 < 1$. Solution of the stated problem was performed in the following manner: stability problem was reduced in Section 1 to solving the spectral problem; solution of the latter made use of the simplicity of the eigenvalue of the steady state problem (Section 2): spectrum of the steady state problem was investigated in Section 3 and in Section 4, theoretical basis was given for the method of solution.

1. We shall seek the solution of the Navier-Stokes equation for the stream function

$$\frac{\partial \Delta\psi}{\partial t} + \psi_y \Delta\psi_x - \psi_x \Delta\psi_y - \nu\Delta^2\psi = \gamma \cos y + \varepsilon f(y, t) \qquad (1.1)$$

periodic in x, y and t and with periods $2\pi/\alpha_0$, 2π and T respectively. We shall restrict the arbitrary constant appearing in the definition of the stream function thus

$$\int_0^T \int_\Omega \psi(x, y, t)\, dx\, dy\, dt = 0, \qquad \Omega = \{|x| \leqslant \pi/\alpha_0, \ \ |y| \leqslant \pi\}$$

and in the following we shall assume that all functions are periodic in x and y and that their periods are $2\pi/\alpha_0$ and 2π. Moreover, we shall assume $f(y, t)$ to be periodic in time with the period T and to satisfy

$$\int_0^T f(y, t)\, dt = 0$$

Let $f(y, t)$ be of the form

$$f(y, t) = \gamma \cos y \left(\sin \omega t + \frac{\omega}{\nu} \cos \omega t\right)$$

Then (1.1) has the following solution

$$\psi_0(y, t) = -\frac{\gamma}{\nu} \cos y\, (1 + \varepsilon \sin \omega t) \tag{1.2}$$

Let us examine its stability. If we put, in (1.1), $\psi(x, y, t) = \psi_0(y, t) + \Phi(x, y, t)$. then $\Phi(x, y, t)$ satisfies

$$\frac{\partial \Delta\Phi}{\partial t} + \Phi_y \Delta\Phi_x - \Phi_x \Delta\Phi_y + \frac{\gamma}{\nu} \sin y\, (1 + \varepsilon \sin \omega t)\, (\Delta\Phi_x + \Phi_x) - \nu\Delta^2\Phi = 0 \tag{1.3}$$

We should note that if $\alpha_0 \geq 1$, then the periodic solution is unconditionally stable (i. e. stable under any perturbations $\Phi(x, y, t)$ and for any values of parameters ν, γ and ϵ). Proof of this follows the lines of the proof in [2] for the steady state solution.

Next we shall solve the linearized problem

$$\frac{\partial \Delta\Phi}{\partial t} + \frac{\gamma}{\nu} \sin y\, (1 + \varepsilon \sin \omega t)\, (\Delta\Phi_x + \Phi_x) - \nu\Delta^2\Phi = 0 \tag{1.4}$$

In analogy with the Folke's method used in ordinary differential equations, we shall seek the solution of (1.4) in the form

$$\Phi(x, y, t) = e^{\sigma t} \varphi(x, y, t)$$

where $\varphi(x, y, t)$ is periodic in t, with period $T = 2\pi/\omega$. Solution $\psi_0(y, t)$ will be unstable if at least one eigenvalue is found, which has a positive real part. Thus, the problem in stability is reduced to the spectral problem

$$A\varphi \equiv \frac{\partial \Delta\varphi}{\partial t} + \sigma\Delta\varphi + \frac{\gamma}{\nu} \sin y\, (1 + \varepsilon \sin \omega t)\, (\Delta\varphi_x + \varphi_x) - \nu\Delta^2\varphi = 0 \tag{1.5}$$

2. We shall now construct the solution of the spectral problem. Let us introduce a Hilbert space H_2 as the closureof the set of smooth periodic functions satisfying the conditions

$$u(-x, -y) = u(x, y), \qquad \int_\Omega u(x, y)\, dx\, dy = 0$$

on the norm, generated by the scalar product

$$(u, v)_{H_2} = \int_\Omega \Delta u \Delta v\, dx\, dy$$

By H_2' we shall denote the Hilbert space of functions of x, y and t belonging to H_2 at almost all t and periodic in t with period T. Scalar product in H_2' is given by

$$(u, v)_{H'_2} = \int_0^T (u, v)_{H_2} dt$$

and by H_2^α (where α is any positive number) we shall denote the subspace of H_2' consisting of functions of the type

$$e^{i\alpha x} g(y, t) + e^{-i\alpha x} g^*(y, t)$$

Here function $g(y, t)$ is periodic in y with the period equal to 2π. Space H_2' decomposes into a simple sum of subspaces $H_2^{k\alpha_0}$ $(k = 1, 2, \ldots)$. Each of these subspaces is invariant with respect to the operater A, hence the investigation of the spectrum of (1.5) reduces to its investigation in the spaces $H_2^\alpha(\alpha = k\alpha_0, k = 1, 2, \ldots)$.

We shall show, how, beginning with the eigenvalue σ_0 and the corresponding eigenfunction $\varphi_0(x, y) \in H_2^\alpha$, and assuming that σ_0 is simple, we can find the eigenvalues and eigenfunctions of (1.5). We shall seek the unknowns σ and $\varphi(x, y, t) \in H_2^\alpha$ in the form of series in ϵ

$$\sigma = \sum_{k=0}^{\infty} \sigma_k \varepsilon^k, \qquad \varphi(x, y, t) = \sum_{k=0}^{\infty} \varphi_k \varepsilon^k \tag{2.1}$$

convergence of which will be proved in Section 4. Inserting (2.1) into (1.5) and equating the coefficients of like powers of ϵ, we obtain a set of equations defining σ_k and φ_k. We shall show, how they can be successively determined.

$$\frac{\partial \Delta\varphi_0}{\partial t} + \sigma_0 \Delta\varphi_0 + \frac{\Upsilon}{\nu} \sin y \frac{\partial}{\partial x} (\Delta\varphi_0 + \varphi_0) - \nu\Delta^2\varphi_0 = 0 \tag{2.2}$$

which corresponds to the stationary case and was discussed in [1], gives σ_0 and φ_0. Unknowns σ_1 and φ_1 are found from Equation

$$\frac{\partial \Delta\varphi_1}{\partial t} + \sigma_0 \Delta\varphi_1 + \frac{\Upsilon}{\nu} \sin y \frac{\partial}{\partial x} (\Delta\varphi_1 + \varphi_1) - \nu\Delta^2\varphi_1 =$$
$$= -\sigma_1 \Delta\varphi_0 - \frac{\Upsilon}{\nu} \sin y \sin \omega t \frac{\partial}{\partial x} (\Delta\varphi_0 + \varphi_0) \tag{2.3}$$

We shall seek the function $\varphi_1(x, y, t)$ in the form

$$\varphi_1(x, y, t) = u_1(x, y) + e^{i\omega t} v_1(x, y) + e^{-i\omega t} v_1^*(x, y)$$

Then u_1 will be given by

$$Lu_1 \equiv \sigma_0 \Delta u_1 + \frac{\Upsilon}{\nu} \sin y \frac{\partial}{\partial x} (\Delta u_1 + u_1) - \nu\Delta^2 u_1 = -\sigma_1 \Delta\varphi_0 \tag{2.4}$$

which has a solution, provided that the necessary condition of orthogonality

$$\int_\Omega (-\sigma_1 \Delta\varphi_0) \tau_0 \, dx \, dy = 0$$

is fulfilled. Here τ_0 is a solution of a conjugate equation

$$L^*\tau_0 \equiv \sigma_0 \Delta\tau_0 - \frac{\Upsilon}{\nu} (1 + \Delta) \frac{\partial}{\partial x} (\varphi \sin y) - \nu\Delta^2\varphi = 0$$

Since the eigenvalue σ_0 is simple, we have

$$\int_\Omega \Delta\varphi_0 \tau_0 \, dx \, dy \neq 0$$

Then $\sigma_1 = 0$, while $u_1(x, y)$ can be found from the equation defining $\varphi_0(x, y)$. Function $v_1(x, y)$ is given by

$$(\sigma_0 + i\omega)\,\Delta v_1 + \frac{\gamma}{\nu}\sin y \frac{\partial}{\partial x}(\Delta v_1 + v_1) - \nu\Delta^2 v_1 = -\frac{1}{2i}\frac{\gamma}{\nu}\sin y \frac{\partial}{\partial x}(\Delta\varphi_0 + \varphi_0) \quad (2.5)$$

Authors of [1] have shown that the eigenvalue σ is real when $\mathrm{Re}\,\sigma \geq 0$, consequently $\sigma_0 + i\omega$ is not an eigenvalue and (2.5) has a solution. Function $v_1(x, y)$ should be sought in form of a Fourier series

$$v_1(x, y) = e^{i\alpha x}\sum_{n=-\infty}^{\infty} a_n e^{iny} + e^{-i\alpha x}\sum_{n=-\infty}^{\infty}(-1)^n a_n e^{iny} \qquad (a_{-n} = (-1)^n a_n)$$

Then we have, for a_n, an infinite nonhomogeneous system of the type

$$c_n a_n + a_{n-1} - a_{n+1} = b_n \qquad (n=0,\ \pm 1, \ldots),$$

which can be solved approximately. After that, we find σ_2 and φ_2 from

$$\frac{\partial\Delta\varphi_2}{\partial t} + L\varphi_2 = -\sigma_2\Delta\varphi_0 - \frac{\gamma}{\nu}\sin y \sin\omega t \frac{\partial}{\partial x}(\Delta\varphi_1 + \varphi_1)$$

We seek φ_2 in the form

$$\varphi_2(x, y, t) = u_2(x, y) + e^{i\omega t} v_2(x, y) + e^{-i\omega t} v_2^*(x, y)$$

function $u_2(x, y)$ is given by

$$Lu_2 = -\sigma_2\Delta\varphi_0 + \frac{\gamma}{\nu}\sin y\ \mathrm{Im}\left[\frac{\partial}{\partial x}(\Delta v_1 + v_1)\right]$$

and the necessary condition for its solution to exist, is

$$\int_\Omega \left\{-\sigma_2\Delta\varphi_0 + \frac{\gamma}{\nu}\sin y\,\mathrm{Im}\left[\frac{\partial}{\partial x}(\Delta v_1 + v_1)\right]\right\}\tau_0\,dx\,dy = 0$$

This yields σ_2 which, in general, is not zero. Remaining unknowns σ_k and φ_k can be found in a similar manner, and all equations encountered are already familiar.

3. Let us now examine the spectrum of the steady state problem, first noting that if the eigenvalue $\sigma_0 \geq 0$ of the problem (2.2) is simple over the class of steady state solutions, then it will be simple over the class of periodic solutions. Indeed, let us seek a solution of (2.2) periodic in t with period T, in form of a Fourier series

$$\varphi_0(x, y, t) = \sum_{k=0}^{\infty} u_k(x, y)\, e^{ik\omega t}$$

Functions $u_k(x, y)$ satisfy

$$(\sigma_0 + ik\omega)\,\Delta u_k + \frac{\gamma}{\nu}\sin y \frac{\partial}{\partial x}(\Delta u_k + u_k) - \nu\Delta^2 u_k = 0 \qquad (k = 0, 1, \ldots)$$

But it was shown in [1] that $\sigma_0 \geq 0$ can be an eigenvalue of this problem only when $k = 0$. Therefore, the absence of associated vectors in the class of periodic solutions follows from their absence from the class of steady state solutions.

1. We shall now show that the positive eigenvalue σ_0 is unique and simple in every subspace $H_2^\alpha (0 < \alpha < 1)$

We shall seek the eigenfunctions $\varphi_0(x, y) \in H_2^\alpha$ of the equation $L\varphi_0 = 0$, in the form

$$\varphi_0(x, y) = e^{i\alpha x}\sum_{n=-\infty}^{\infty} c_n e^{iny} + e^{-i\alpha x}\sum_{n=-\infty}^{\infty}(-1)^n c_n e^{iny} \qquad (3.1)$$

where the coefficients c_n satisfy the condition $c_{-n} = (-1)^n c_n$.

Then, as shown in [1], the eigenvalue σ_0 can be found from

$$-\frac{a_0}{2}=\frac{1}{a_1}+\frac{1}{a_2+\dots}\equiv f(\mu,\lambda,\alpha) \qquad (3.2)$$

in which the following notation is used:

$$\mu=\frac{\sigma_0}{\nu},\qquad \lambda=\frac{\gamma}{\nu^2},\qquad a_n=\frac{2}{\lambda}\frac{(\alpha^2+n^2)(\alpha^2+n^2+\mu)}{\alpha(\alpha^2-1+n^2)}\qquad (n=0,1,\dots)$$

We easily see that if $\alpha\geq 0$, then (3.2) has neither positive, nor zero roots.

Lemma 3.1. Function $\lambda f(\mu,\lambda,\alpha)$ increases monotonely in λ.

Proof. We have

$$\lambda f(\mu,\lambda,\alpha)\equiv\frac{1}{\lambda^{-1}a_1}+\frac{1}{\lambda a_2}+\dots$$

When λ increases, odd terms of the above continued fraction decrease, while the even terms remain unchanged, which proves the lemma.

Lemma 3.2. Function $a_0^{-1}f(\mu,\lambda,\alpha)$ increases monotonely in μ ($\mu>0$).

Proof. We have

$$a_0^{-1}f(\mu,\lambda,\alpha)\equiv\frac{1}{a_0a_1}+\frac{1}{a_0^{-1}a_2}+\dots$$

Odd terms are of the form

$$a_0a_n=\frac{4}{\lambda^2}\frac{(\alpha^2+\mu)\left[(\alpha^2+n^2)^2+\mu(\alpha^2+n^2)\right]}{(\alpha^2-1)(\alpha^2-1+n^2)}<0\qquad (n=1,3,\dots)$$

Obviously, they decrease with increasing μ ($\mu>0$). Even terms are of the form

$$a_0^{-1}a_n=\frac{(\alpha^2+n^2)(\alpha^2+n^2+\mu)(\alpha^2-1)}{\alpha^2(\alpha^2-1+n^2)(\alpha^2+\mu)}\qquad (n=2,4,\dots)$$

We easily see that the derivative $\partial(a_0^{-1}a_n)/\partial\mu>0$, hence the continued fraction increases in $\mu>0$.

Lemma 3.3. If $0<\alpha<1$ and $\lambda\geq\lambda_0$ where λ_0 is a solution of (3.2) when $\mu=0$, then Equation (3.2) has a single root $\mu\geq 0$.

Proof. (a). If $\mu\to\infty$, then $-\frac{1}{2}a_0\to+\infty$. The estimate

$$f\leqslant\frac{1}{a_1}=\frac{\lambda}{2}\frac{\alpha^3}{(\alpha^2+1)(\alpha^2+1+\mu)}$$

holds for the function $f(\mu,\lambda,\alpha)$ and, as $\mu\to+\infty$, we have

$$-{}^1/_2a_0>f(\mu,\lambda,\alpha) \qquad (3.3)$$

b). We shall show that for small μ the opposite inequality holds. In [2] the existence was proved of such $\lambda=\lambda_0$, for which $\mu=0$ was a root of (3.2), i.e.

$$\frac{\alpha^3}{1-\alpha^2}=\lambda_0 f(0,\lambda_0,\alpha)$$

Function $\lambda f(\mu,\lambda,\alpha)$ increases monotonously in λ (Lemma 3.1), therefore at $\lambda\geq\lambda_0$ and small values $\mu\geq 0$, we have

$$-\frac{a_0}{2}\leqslant f(\mu,\lambda,\alpha) \qquad (3.4)$$

Comparing the estimates (3.3) and (3.4) we can deduce the existence of a root $\mu\geq 0$ of (3.2), and its uniqueness for a fixed α follows from Lemma 3.2.

In order to establish definitely the simplicity of the eigenvalue in the space H_2, we must prove the absence of associated vectors which, in turn, requires that Equation $L\varphi=-\Delta\varphi_0$ has no solution. It can have a solution only if the condition

$$\theta\equiv-\int_\Omega\Delta\varphi_0\tau_0\,dx\,dy=0$$

where τ_0 is the solution of the conjugate equation, holds. We shall show now, that $\theta>0$. Eigenfunctions $\varphi_0(x,y)$ have the form (3.1), while those of the conjugate equation $\tau_0(x, y)$, have the form

$$\tau_0(x, y) = e^{i\alpha x} \sum_{n=-\infty}^{\infty} d_n e^{iny} + e^{-i\alpha x} \sum_{n=-\infty}^{\infty} (-1)^n d_n e^{iny}, \quad d_{-n} = (-1)^n d_n$$

It can be confirmed directly that the coefficients c_n and d_n are connected by the relation $d_n = (-1)^{n-1}(\alpha^2 + n^2 - 1)\, c_n$. In this case we obtain

$$\theta = 2\,|\,\Omega\,| \sum_{n=-\infty}^{\infty} (-1)^{n-1} (\alpha^2 + n^2)(\alpha^2 + n^2 - 1)\, c_n^2 \tag{3.5}$$

We shall establish another equation to show that $\theta > 0$. Multiplying (2.2) by the function $\Delta\varphi_0 + \varphi_0$ and integrating over Ω, we obtain

$$\sum_{n=-\infty}^{\infty} (\alpha^2 + n^2)^2 (\alpha^2 + n^2 - 1)\, c_n^2 + \mu \sum_{n=-\infty}^{\infty} (\alpha^2 + n^2)(\alpha^2 + n^2 - 1)\, c_n^2 = 0 \tag{3.6}$$

which, multiplied by $(\alpha^2 + \mu)^{-1}$ and added to (3.5), yields

$$\theta = 2\,|\,\Omega\,| \sum_{n\neq 0} (\alpha^2 + n^2)(\alpha^2 + n^2 - 1)\left[(-1)^{n-1} + 1 + \frac{n^2}{\alpha^2 + \mu}\right] c_n^2 > 0$$

2. If $\frac{1}{2} \leq \alpha_0 < 1$, then $\mu(\alpha_0)$ is a unique simple eigenvalue in H_2'.
Indeed, the eigenvalue $\mu(\alpha_0)$ is simple in the space $H_2^{\alpha_0}$, while other spaces $H_2^{k\alpha_0}$ $(k \geqslant 2)$ contain no eigenfunctions corresponding to $\mu(\alpha_0)$.

4. We shall now give the proof of our method of solving the spectrum problem. Let us show that, for sufficiently small ε, series (2.1) converge. If we introduce into (1.5) the following operator

$$B \equiv \frac{\partial}{\partial t}\Delta - \nu\Delta^2$$

then (1.5) will become

$$\varphi + K\varphi + \varepsilon K_1\varphi = \sigma K_2\varphi \tag{4.1}$$

$$K \equiv \frac{\gamma}{\nu} B^{-1} \sin y \frac{\partial}{\partial x}(\Delta + 1), \qquad K_1 \equiv \sin \omega t K, \qquad K_2 \equiv -B^{-1}\Delta$$

Operators K, K_1 and K_2 operating in H_2' were initially defined on smooth functions and then extended by virtue of continuity, over the whole space H_2'.

If, for example, Fourier series in x, y, t are used to invert the operator B, then we can easily confirm that the operators K, K_1 and K_2 are fully continuous in H_2'.

When $\varepsilon = 0$, the eigenvalue σ_0 and the eigenfunction $\varphi_0(x, y)$ are known [1] and σ_0 is a simple eigenvalue. Let us normalize the eigenfunctions of (4.1) by means of the condition $(\varphi, \tau_0)_{H_2'} = 1$. We shall seek the unknowns in the form

$$\varphi(x, y, t) = \varphi_0(x, y) + u(x, y, t), \qquad \sigma = \sigma_0 + \mu$$

Then, the unknowns $u(x, y, t)$ and μ should satisfy the equation

$$Du \equiv u + Ku - \sigma_0 K_2 u = \mu K_2 u + \mu K_2 \varphi_0 - \varepsilon K_1 u - \varepsilon K_1 \varphi_0 \tag{4.2}$$

and condition

$$(u, \tau_0)_{H_2'} = 0.$$

should hold for $u(x, y, t)$.

In the following steps of the proof we make use of the equation of branching [3].

Equation (4.2) has a solution if and only if the condition of orthogonality

$$(\mu K_2 u + \mu K_2 \varphi_0 - \varepsilon K_1 u - \varepsilon K_1 \varphi_0,\ \tau_0)_{H_2'} = 0 \tag{4.3}$$

is fulfilled.

Let us introduce the operator $R\,(H_2' \to H_2')$ such, that $DRf = f$ if $(f, \tau_0) = 0$.

Then, from (4. 2) we have

$$u - \mu RK_2 u + \varepsilon RK_1 u = \mu RK_2\varphi_0 - \varepsilon RK_1\varphi_0 \qquad (4.4)$$

Applying now to (4. 4) the contraction and reflection transformations, we conclude, that with ϵ and μ sufficiently small to satisfy

$$\| \mu RK_2 - \varepsilon R K_1 \|_{H_2'} < 1 \qquad (4.5)$$

there exists a unique solution $u(x, y, t)$ of Equation (4. 4)

$$u(x, y, t) = (1 - \mu RK_2 + \varepsilon RK_1)^{-1}(\mu RK_2\varphi_0 - \varepsilon RK_1\varphi_0) \qquad (4.6)$$

From the latter we see that $u(x, y, t)$ can be expanded into series in powers of ϵ and μ. Inserting (4. 6) into (4. 3) we obtain $F(\mu, \epsilon) = 0$ where F is an analytic function and

$$\frac{\partial F(0, 0)}{\partial \mu} = (K_2\varphi_0, \tau_0)_{H'_2} \neq 0$$

Putting (4. 6) into (4. 3) we find that $\mu(\epsilon)$ can be expressed in terms of a series in ϵ. It can easily be checked that if $\mu(\epsilon)$ in its expanded form is substituted into (4. 6), then the function $u(x, y, t)$ will satisfy (4. 4).

In conclusion we note that the method of investigation of stability of periodic soluti^r given in Section 2 allows an additional conclusion to be drawn on the influence of smal periodic forces on the stability of the steady state solution. Thus, if $\sigma_0 = 0$ and Re $\sigma_2 > 0$, then the solution moves from the neutral, into the unstable region.

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